Hyperfinite-operational Approach to the Problem of Time Reversibility of Quantum Mechanics

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Abstract

This paper outlines a mathematical framework of quantum probability in which the time asymmetry in describing measuring processes is avoided. The main objects of the framework are hyperfinite operations, which are constructed by using nonstandard analysis and the operational approach by Davies and Lewis. Then the notions of Bayesian conditional probability are defined, and Bayes-type theorems in terms of the probability are showed.

1 Introduction

Belinfante(1) argued the problem of retrodictions in quantum physics. He insists: conventional quantum theory has only predictive concepts, e.g., predictive probability and state (he rather prefers the word "postdictive," but its implication is similar to "retrodictive"). However, to complete the formalism of quantum mechanics we need a satisfactory treatment of retrodictive concepts.

This proposal leads to his program of "time-symmetric quantum theory" including quantum measurement theory. This program seems to be of great interest in two respects; one is theoretical, and the other is philosophical.

The first point is coherence of the formalism of quantum measurement theory (usually nonrelativistic) and that of relativistic quantum theory(1,2). Roughly speaking, the way in which quantum measurement theory treats the time coordinate and the way in which it treats the space coordinates are radically different. A great part of the difference is caused by the time asymmetry of quantum measurement theory.

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The second point is the problem of interpretation of state reduction, e.g., Schrödinger's cat paradox. The time asymmetry appears mainly in descriptions of state reduction[1]. Ozawa(5) gave a new formulation to the problem in the framework of C^*- dynamical systems. He proved that in the framework we cannot describe the measuring interaction between a microscopic system and a macroscopic apparatus as long as the time reversibility of the dynamics of isolated system is assumed.

The present paper proposes a mathematical framework of time-symmetric quantum physics. This framework uses two tools: the operational approach to quantum probability by Davies and Lewis(6,7), and the method of nonstandard analysis originated by A.Robinson(8). Our idea of the resolution of time asymmetry is similar to that of Belinfante's; conventional quantum measurement theory has only predictive concepts, so it has time asymmetry. Therefore, we should add retrodictive concepts to it. We shall introduce two types of Bayesian conditional probability (predictive and retrodictive) in section 4. These two types of probability will be related by Bayes-type theorems.

2 Nonstandard Analysis

This section briefly outlines the theory of nonstandard analysis(5). Let X be a set and $\mathcal{P}(X)$ the power set of X, that is, the set of all subsets of X. The *superstructure over* X, denoted by $V(X)$, is defined by the following recursion:

$$V_0(X) = X, V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)),$$
$$V(X) = \bigcup_{n \in \mathbf{N}} V_n(X),$$

where $\mathbf{N}$ is the set of natural numbers. Let us regard any element of X as a nonset here; hence $x \in V(X)$ is a set iff $x \in V(X) \setminus X$. Let $\mathbf{C}$ be the set of complex numbers. $V(\mathbf{C})$ contains all the structures that we use in quantum physics; for instance, separable Hilbert space $\mathcal{H}$ is in $V(\mathbf{C})$.

$V(X)$ is called a *nonstandard extension of* $V(\mathbf{C})$ if there exists a map $\star : V(\mathbf{C}) \longrightarrow V(X)$ satisfying the following conditions:

(1) $\star$ is an injective mapping from $V(\mathbf{C})$ to $V(X)$,

(2) ${}^\star\mathbf{C} = X$,

(3) (Transfer Principle) Let ϕ be a sentence in terms of $V(\mathbf{C})$, and ${}^\star\phi$ be the sentence "transfered" from ϕ by mapping $\star$. ϕ is true iff ${}^\star\phi$ is true.

[1]Sometimes "state reduction" is used to refer a transformation from a pure state to a mixed state(4), sometimes to refer state transformation $|\psi\rangle \to P|\psi\rangle$ where $|\psi\rangle$ is a vector in Hilbert space $\mathcal{H}$ and P is a projector on $\mathcal{H}$. Both of them imply time asymmetry though those usage must be distinguished.

Transfer Principle needs more explanation. A sentence in terms of $V(\mathbf{C})$ is constructed from the symbols for logical connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, quantifiers $\forall, \exists$, individual variables $x_1, x_2, \ldots$, two predicates $=, \in$, parentheses $($, $)$, and elements of $V(\mathbf{C})$.

We will consider an example. Let $\mathbf{R}$ denote the set of real numbers. Define $G_< \in V(\mathbf{C})$ by $G_< = \{(x, y) |\, x, y \in \mathbf{R}, x < y\}$, where (x, y) is identified as $\{\{x\}, y\}$.

$$(\forall x)(\forall y)(x \in \mathbf{R} \wedge y \in \mathbf{R} \wedge (x, y) \in G_< \Rightarrow (\exists z)(z \in \mathbf{R}$$

$$\wedge (x, z) \in G_< \wedge (x, y) \in G_<))$$

is a sentence in terms of $V(\mathbf{C})$ because $\mathbf{R}, G_< \in V(\mathbf{C})$. Let ϕ denote this sentence. ϕ means that $\mathbf{R}$ is dense, and hence ϕ is true. The "transfered" sentence ${}^\star\phi$ is as follows:

$$(\forall x)(\forall y)(x \in^\star \mathbf{R} \ \wedge \ y \in^\star \mathbf{R} \ \wedge \ (x, y) \in^\star G_< \Rightarrow (\exists z)(z \in^\star \mathbf{R}$$

$$\wedge \, (x, z) \in^\star G_< \ \wedge \ (x, y) \in^\star G_<))$$

By Transfer Principle, ${}^\star\phi$ is true (${}^\star\mathbf{R}$ is called $\star$-dense).

$u \in V(X)$ is called *standard* if there is $x \in V(\mathbf{C})$ such that $u =^\star x$, and called *internal* if there is $x \in V(\mathbf{C})$ such that $u \in^\star x$. ${}^\star V(\mathbf{C})$ is the set of all internal sets. Let A and B be internal sets. Function $f : A \longrightarrow B$ is called *internal* if the graph of f, that is, $\{(x, f(x)) | x \in A\}$ is internal. $V(X)$ is called a *countably saturated extension of* $V(\mathbf{C})$ if it satisfies the following condition:

(Saturation Principle) If countable sequence of internal sets $A_j \in V(X) \setminus X$ satisfies

$$\bigcap_{j=1}^{k} A_j \neq \phi \, (k = 1, 2, \ldots)$$

then

$$\bigcap_{j=1}^{\infty} A_j \neq \phi.$$

Let $A \in V(\mathbf{C}) \setminus \mathbf{C}$. From the Saturation Principle, we can show that if $A \in V(\mathbf{C})$ is an infinite set, ${}^s A$ defined by

$${}^s A = \{{}^\star a |\, a \in A\}$$

is a proper subset of ${}^\star A$. Thus ${}^\star A$ is an extended structure of A. ${}^\star A$ is called the *nonstandard extension of* A. By renaming the elements of X, we can assume without loss of generality that $\mathbf{C}$ is a subset of ${}^\star\mathbf{C}$ and ${}^\star x = x$ for each $x \in \mathbf{C}$.

Given any subset $U \subseteq V(\mathbf{C})$, define ${}^\star U$ by

$${}^\star U = \bigcup_{n \in \mathbf{N}} {}^\star(U \cap V_n(\mathbf{C})).$$

Let $F(\mathbf{C}) \subseteq V(\mathbf{C})$ be the set of finite sets, i.e.,

$$F(\mathbf{C}) = \{A \in V(\mathbf{C}) \setminus \mathbf{C} | \, A \text{ is a finite set}\}.$$

An element of ${}^\star F(\mathbf{C})$ is called a *hyperfinite set*. Any element of ${}^\star\mathbf{C}\,({}^\star\mathbf{R})$ is called a *hypercomplex (hyperreal)* number. We assume that ${}^\star\mathbf{R} \subseteq {}^\star\mathbf{C}$. ${}^\star\mathbf{C}$ is a proper extension field of $\mathbf{C}$, and ${}^\star\mathbf{R}$ is an ordered extension of $\mathbf{R}$. An element of ${}^\star\mathbf{N}\,({}^\star\mathbf{Z})$ is called a *hypernatural number (hyperinteger)*. It is shown that if A is a hyperfinite set (i.e. $A \in {}^\star F(\mathbf{C})$) then there is an initial segment $J = \{n \in {}^\star\mathbf{N} | \, n \leq j\}$ for some $j \in {}^\star\mathbf{N}$ and a one-to-one, onto internal mapping $f : J \longrightarrow A$. Thus, we will often write a hyperfinite set A as $A = \{a_1, a_2, ..., a_j\}$, where $a_k = f(k), k \in J$.

A hypercomplex number x is called *infinite* if $|x| > n$ for any $n \in \mathbf{N}$, *finite*, $|x| < \infty$, if there is some $n \in \mathbf{N}$ such that $|x| < n$, and *infinitesimal* if $|x| < \frac{1}{n}$ for any $n \in \mathbf{N}$.

For any $x, y \in {}^\star\mathbf{C}$, we will write $x \approx y$ if $|x - y|$ is infinitesimal. For any finite hyperreal number x, there is a unique real number r such that ${}^\star r \approx x$; this r is called the *standard part of* of x and denoted by ${}^\circ x$.

Any function f from A to B is extended to an internal function ${}^\star f$ from ${}^\star A$ to ${}^\star B$. A sequence $a_n \in \mathbf{C}\,(n \in \mathbf{N})$ is extended to an internal sequence ${}^\star a_\nu \in {}^\star\mathbf{C}\,(\nu \in {}^\star\mathbf{N})$, so that $\lim_{n\to\infty} a_n = a$ if and only if ${}^\star a_\nu \approx a$ for all $\nu \in {}^\star\mathbf{N} \setminus \mathbf{N}$.

Let $\mathcal{A}$ be an internal normed linear space with norm $\| \cdot \|$. The *principal galaxy* $\mathrm{fin}(\mathcal{A})$ and the *principal monad* $\mu(0)$ are defined by

$$\mathrm{fin}(\mathcal{A}) = \{x \in \mathcal{A} | \; \| x \| < \infty\}$$

$$\mu(0) = \{x \in \mathcal{A} | \; \| x \| \approx 0\}$$

Both of them are linear spaces over $\mathbf{C}$. The *nonstandard hull of* $\mathcal{A}$ is the quotient linear space $\hat{\mathcal{A}} = \mathrm{fin}(\mathcal{A})/\mu(0)$ equipped with the norm given by

$$\| {}^\circ x \| = {}^\circ \| x \|$$

for all $x \in \mathrm{fin}(\mathcal{A})$, where ${}^\circ x = x + \mu(0)$. It is shown by the Saturation Principle that $\hat{\mathcal{A}}$ is a Banach space(Ref.(9), p.155).

Let ν be an infinite hypernatural number, i.e., $\nu \in {}^\star\mathbf{N} \setminus \mathbf{N}$, and ${}^\star\mathbf{C}^\nu$ the ν-dimensional internal inner product space with the natural inner product and the internal norm $\| \cdot \|$ derived by the inner product. Let $\mathbf{M} = {}^\star M(\nu)$ be the internal algebra of $\nu \times \nu$ matrices over ${}^\star\mathbf{C}$. Naturally, $\mathbf{M}$ acts on ${}^\star\mathbf{C}^\nu$ as the internal linear operators, and let p_∞ be the operator norm on $\mathbf{M}$, i.e., $p_\infty(A) = \sup\{\| A\xi \| \mid \| \xi \| \leq 1, \xi \in {}^\star\mathbf{C}^\nu\}$. Denote by A^* the adjoint of $A \in \mathbf{M}$. Let τ be the internal normalized trace on $\mathbf{M}$, i.e.,

$$\tau(A) = \frac{1}{\nu}\sum_{i=1}^{\nu} A_{ii}$$

for $A = (A_{ij}) \in \mathbf{M}$. Then τ defines an internal inner product $(\cdot|\cdot)$ on $\mathbf{M}$ by $(A|B) = \tau(A^*B)$, for $A, B \in \mathbf{M}$. Its derived norm called the normalized

Hilbert-Schmidt norm is denoted by p_2, i.e., $p_2(A) = \tau(A^*A)^{1/2}$ for $A \in \mathbf{M}$. Denote by $(\mathbf{M}, p_\infty)$ and $(\mathbf{M}, p_2)$ the normed linear spaces equipped with these respective norms. The principal galaxies $\mathrm{fin}_\infty(\mathbf{M})$ of $(\mathbf{M}, p_\infty)$ and $\mathrm{fin}_2(\mathbf{M})$ of $(\mathbf{M}, p_2)$ are defined as follows:

$$\mathrm{fin}_\infty(\mathbf{M}) = \{A \in \mathbf{M} |\, p_\infty(A) < \infty\}$$

$$\mathrm{fin}_2(\mathbf{M}) = \{A \in \mathbf{M} |\, p_2(A) < \infty\}$$

The principal monads $\mu_\infty(0)$ of $(\mathbf{M}, p_\infty)$ and $\mu_2(0)$ of $(\mathbf{M}, p_2)$ are defined as follows:

$$\mu_\infty(0) = \{A \in \mathbf{M} |\, p_\infty(A) \approx 0\}$$

$$\mu_2(0) = \{A \in \mathbf{M} |\, p_2(A) \approx 0\}.$$

The nonstandard hull $\hat{\mathbf{M}}_2 = \mathrm{fin}_2(\mathbf{M})/\mu_2(0)$ turns out to be a Hilbert space with inner product $< \cdot | \cdot >$ and norm $\| \cdot \|_2$ defined by

$$< A + \mu_2(0) | B + \mu_2(0) >= {}^\circ(A|B)$$

and

$$\| A + \mu_2(0) \|_2 = p_2(A).$$

for $A, B \in \mathrm{fin}_2(\mathbf{M})$.

The nonstandard hull $\hat{\mathbf{M}}_\infty = \mathrm{fin}_\infty(\mathbf{M})/\mu_\infty(0)$ of $(\mathbf{M}, p_\infty)$ turns out to be a C^*-algebra equipped with norm $\hat{p}_\infty$ defined by

$$\hat{p}_\infty(A + \mu_\infty(0)) = {}^\circ p_\infty(A)$$

for $A \in \mathrm{fin}_\infty(\mathbf{M})$.

Hinokuma and Ozawa[10] showed that another quotient space $\hat{\mathbf{M}}$ defined by

$$\hat{\mathbf{M}} = \mathrm{fin}_\infty(\mathbf{M})/(\mu_2(0) \cap \mathrm{fin}_\infty(\mathbf{M})).$$

is a von Neumann algebra of type II_1 factor.

3 Operation

This section briefly reviews the notion of operation and instrument in the sense of Davies ,Lewis, Srinivas and Ozawa[6,7,11,12].

The state space V of a Hilbert space $\mathcal{H}$ is defined as the Banach space $\mathcal{T}_s(\mathcal{H})$ of trace class operators on $\mathcal{H}$ with trace norm $\| \cdot \|_{\mathrm{tr}}$ $(= \mathrm{tr}\,| \cdot |)$. The states are defined as the non-negative trace class operators V^+ of trace one, elsewhere called density matrix.

If V is the state space of Hilbert space $\mathcal{H}$, an *operation* on V is defined as a positive linear map $T : V \to V$ which also satisfies

$$0 \leq \mathrm{tr}(T\rho) \leq \mathrm{tr}\,\rho.$$

for all $\rho \in V^+$. The concept of operation is thought of as an extension of that of projective observations. Let P be a projector on $\mathcal{H}$ and ρ a state. The probability that the "proposition" P is observed by a projective observation is $\mathrm{tr}(P\rho)$ and the state at the instant after the observation is $P\rho P/\,\mathrm{tr}(P\rho)$. If T denote the map $T : \rho \mapsto P\rho P$, then the probability is written as $\mathrm{tr}(T\rho)$ and the state as $T\rho/\,\mathrm{tr}(T\rho)$. T is an example of an operation.

An *instrument* is defined as a mapping

$$I : B(R) \times V \to V$$

where $B(R)$ is the set of all Borel sets of a value space R (usually a real line). The requirements on I for it to define an instrument are

(i) $I(E, \cdot)$ is an operation for all $E \in B(R)$.

(ii)$I(\bigcup_i E_i, \rho) = \sum_i I(E_i, \rho)$ for each countable family $\{E_i\}$ of pairwise disjoint sets of $E_i \in B(R)$.

(iii)$\mathrm{tr}\, I(R, \rho) = \mathrm{tr}\, \rho$.

Davies and Lewis proved that given two instruments I_1 and I_2 on value spaces R_1 and R_2, we can consider their composition $I_2(E_2, I_1(E_1, \rho))$ as a unique instrument I defined on $R_1 \times R_2$.

Ozawa(12) introduced the concept of realizability of an operation, and showed that operation T is realizable iff T is completely positive, that is, for any finite sequence of vectors $\xi_1, ..., \xi_n$ and $\eta_1, ..., \eta_n$,

$$\sum_{i,j=1}^{n} \langle \xi_i | T(|\eta_i\rangle\langle\eta_j|) | \xi_j \rangle \geq 0.$$

Any operation which is not realizable is of no interest from a physical point of view, so we may add complete positivity to the definition of operation .

4 Hyperfinite Operation

Let ${}^\star\mathcal{L}(\mathbf{M})$ denote the set of all internal linear mapping $a : \mathbf{M} \to \mathbf{M}$.

Definition 4.1 Let $a \in {}^\star\mathcal{L}(\mathbf{M})$. $a^{\leftrightarrow}, a^{\updownarrow}, a^{\diamond} \in {}^\star\mathcal{L}(\mathbf{M})$ are defined as follows:

$$a^{\leftrightarrow}(A) = [a(A^*)]^*,$$

$$\mathrm{tr}[B^* a^{\updownarrow}(A)] = \mathrm{tr}[A^* a(B)]$$

for all $A, B \in \mathbf{M}$, and

$$\langle\gamma|a^{\diamond}(|\alpha\rangle\langle\beta|)|\delta\rangle = \langle\gamma|a(|\delta\rangle\langle\beta|)|\alpha\rangle$$

for any pairwise orthogonal vectors $|\alpha\rangle, |\beta\rangle, |\gamma\rangle$, and $|\delta\rangle$.

Suppose $A \in \mathbf{M}$ and $a \in {}^\star\mathcal{L}(\mathbf{M})$. If we represent A as $(A_{\alpha\beta})$, $a(A)$ is represented as

$$\Big(\sum_{\alpha\beta} a^{\alpha\beta}_{\gamma\delta} A_{\alpha\beta}\Big)_{\gamma\delta},$$

and hence a is denoted as

$$(a^{\alpha\beta}_{\gamma\delta}).$$

Under this denotation, we see that

$$(a^{\leftrightarrow})^{\alpha\beta}_{\gamma\delta} = \overline{a^{\beta\alpha}_{\delta\gamma}}$$

$$(a^{\updownarrow})^{\alpha\beta}_{\gamma\delta} = \overline{a^{\gamma\delta}_{\alpha\beta}}$$

$$(a^{\diamond})^{\alpha\beta}_{\gamma\delta} = a^{\delta\beta}_{\gamma\alpha}$$

For $a, b \in {}^\star\mathcal{L}(\mathbf{M})$, let ab denote the composed mapping of b and a. Thus, the following properties are clear:

(i) $a^{\leftrightarrow\leftrightarrow} = a^{\updownarrow\updownarrow} = a^{\diamond\diamond} = a$,

(ii) $a^{\leftrightarrow\diamond} = a^{\diamond\updownarrow}$, $a^{\diamond\leftrightarrow} = a^{\updownarrow\diamond}$,

(iii)$(ab)^{\updownarrow} = b^{\updownarrow}a^{\updownarrow}$, $(ab)^{\leftrightarrow} = a^{\leftrightarrow}b^{\leftrightarrow}$,

(iv) $a^{\updownarrow\leftrightarrow} = a^{\leftrightarrow\updownarrow}$

Definition 4.2 Suppose $a \in {}^\star\mathcal{L}(\mathbf{M})$. a is called *positive* if for some $b \in {}^\star\mathcal{L}(\mathbf{M}), a = b^{\updownarrow}b$.

This definition is an analogy of that of positive matrix; a matrix A is called *positive* if $A = B^*B$ for some matrix B.

Proposition 4.1 Suppose $a \in {}^\star\mathcal{L}(\mathbf{M})$. Following three conditions are equivalent.

(i) a is positive.

(ii) $\mathrm{tr}[A^*a(A)] \geq 0$ for each $A \in \mathbf{M}$.

(iii) $\langle\alpha|a(|\alpha\rangle\langle\beta|)|\beta\rangle \geq 0$ for all vectors $|\alpha\rangle, |\beta\rangle \in {}^\star\mathbf{C}^\nu$.

Proof Evident from the fact that $< A, B >= \mathrm{tr}(A^*B)$ is an inner product on $\mathbf{M}$,and $a^{\updownarrow}$ is the adjoint operator of a on the inner product space $(\mathbf{M}, <, >)$. □

Proposition 4.2 Suppose $a \in {}^\star\mathcal{L}(\mathbf{M})$. Following four conditions are equivalent.

(i) $a^{\diamond}$ is positive.

(ii) $\mathrm{tr}[Aa(B)] \geq 0$ for all positive $A, B \in \mathbf{M}$.

(iii) $a(A)$ is a positive matrix for any positive $A \in \mathbf{M}$.

(iv) $\langle\alpha|a(|\beta\rangle\langle\beta|)|\alpha\rangle \geq 0$ for all $|\alpha\rangle, |\beta\rangle \in {}^\star\mathbf{C}^\nu$.

We see from (iii) that if $a^\diamond$ and $b^\diamond$ are positive, $(ab)^\diamond$ is positive.

Proposition 4.3 Suppese $a \in {}^\star\mathcal{L}(\mathbf{M})$. $a^\diamond$ is positive iff there are $\kappa \in {}^\star\mathbf{N}$ and $M_k \in \mathbf{M}\,(k = 1, \cdots, \kappa)$ such that

$$a(A) = \sum_{k=1}^{\kappa} M_k A M_k^*.$$

Proof If $a(A) = \sum_{k=1}^{\kappa} M_k A M_k^*$, $a^\diamond$ is positive by Proposition 4.2 (iii). Conversely, if $a^\diamond$ is positive, there is $b \in {}^\star\mathcal{L}(\mathbf{M})$ such that $a^\diamond = b^\updownarrow b$, and hence we have

$$a^{\alpha\beta}_{\gamma\delta} = \sum_{\epsilon,\zeta} \overline{(b^\diamond)^{\zeta\alpha}_{\epsilon\gamma}}(b^\diamond)^{\zeta\beta}_{\epsilon\delta}.$$

If $B^\alpha_\beta \in \mathbf{M}\,(\alpha, \beta = 1, \cdots, \kappa)$ are defined by

$$(B^\alpha_\beta)_{ij} = (b^\diamond)^{\alpha i}_{\beta j},$$

then,

$$\begin{aligned}
[a(A)]_{\gamma,\delta} &= \sum_{\alpha,\beta} a^{\alpha\beta}_{\gamma\delta} A_{\alpha\beta} = \sum_{\alpha,\beta,\epsilon,\zeta} \overline{(b^\diamond)^{\zeta\alpha}_{\epsilon\gamma}} A_{\alpha\beta}(b^\diamond)^{\zeta\beta}_{\epsilon\delta} \\
&= \Big(\sum_{\zeta,\epsilon} B^{\zeta *}_\epsilon A B^\zeta_\epsilon\Big)_{\gamma,\delta}.
\end{aligned}$$

Thus $a(A) = \sum_{\zeta,\epsilon} B^{\zeta *}_\epsilon A B^\zeta_\epsilon$. $\square$

Corollary Let $a \in {}^\star\mathcal{L}(\mathbf{M})$. $a^\diamond$ is positive iff a is $\star$- completely positive, i.e., for any hyperfinite sequence of vectors $|\xi_1\rangle, ..., |\xi_\kappa\rangle, |\eta_1\rangle, ..., |\eta_\kappa\rangle \in {}^\star\mathbf{C}^\nu$,

$$\sum_{i,j=1}^{\kappa} \langle\xi_i|T(|\eta_i\rangle\langle\eta_j|)|\xi_j\rangle \geq 0$$

Definition 4.3 Let $a \in {}^\star\mathcal{L}(\mathbf{M})$. Linear functionals $\mathrm{tr}^\updownarrow, \mathrm{tr}^\leftrightarrow$ over ${}^\star\mathcal{L}(\mathbf{M})$ are defined as follows:

$$\mathrm{tr}^\updownarrow a = \sum_{\alpha,\beta} a^{\alpha\beta}_{\alpha\beta},$$

$$\mathrm{tr}^\leftrightarrow a = \sum_{\alpha,\beta} a^{\alpha\alpha}_{\beta\beta}.$$

Evidently, we have $\mathrm{tr}^\updownarrow(ab) = \mathrm{tr}^\updownarrow(ba)$, while $\mathrm{tr}^\leftrightarrow(ab) = \mathrm{tr}^\leftrightarrow(ba)$ does not hold.

Proposition 4.4 Let $I \in \mathbf{M}$ denote the identity matrix.

(i) $\mathrm{tr}^\leftrightarrow(a^\diamond) = \mathrm{tr}^\updownarrow a$.

(ii) $\mathrm{tr}^\updownarrow(a^\diamond) = \mathrm{tr}^\leftrightarrow a$.

(iii) $\mathrm{tr}^{\leftrightarrow} a = \mathrm{tr}\, a(I)$.

(iv) $\mathrm{tr}^{\updownarrow} a = \mathrm{tr}\, a^{\diamond}(I)$.

(v) $\mathrm{tr}^{\leftrightarrow}(ab) = \mathrm{tr}[a^{\updownarrow}(I)^* b(I)]$

Proof (v). Let $\delta_{\alpha\beta}$ be Kronecker's notation.

$$\mathrm{tr}^{\leftrightarrow}(ab) = \sum_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta} \delta_{\alpha\beta} a_{\alpha\beta}^{\gamma\delta} b_{\gamma\delta}^{\epsilon\zeta} \delta_{\epsilon\zeta} = \sum_{\gamma,\delta} \overline{(a^{\updownarrow}(I))_{\gamma,\delta}} (b(I))_{\gamma,\delta}$$

$$= \mathrm{tr}[a^{\updownarrow}(I)^* b(I)]. \quad \square$$

Proposition 4.5 Let $a \in {}^{\star}\mathcal{L}(\mathbf{M})$. Suppose $a^{\diamond}$ be positive. If

$$a(A) = \sum_{k=1}^{\kappa} M_k A M_k^* \; (\kappa \in {}^{\star}\mathbf{N}, M_k \in \mathbf{M})$$

for all $A \in \mathbf{M}$, then,

$$a^{\updownarrow}(A) = \sum_{k=1}^{\kappa} M_k^* A M_k$$

for all $A \in \mathbf{M}$.

Proposition 4.6 Let $a, b \in {}^{\star}\mathcal{L}(\mathbf{M})$.

(i) If $a^{\diamond}$ is positive, $\mathrm{tr}^{\leftrightarrow} a \geq 0$

(ii) If $a^{\diamond}$ and $b^{\diamond}$ are positive and $a(I) \leq I$, then $\mathrm{tr}^{\leftrightarrow}(ab) \leq \mathrm{tr}^{\leftrightarrow} b$.

(iii) If $a^{\diamond}$ and $b^{\diamond}$ are positive, and $a^{\updownarrow}(I) \leq I$, then $\mathrm{tr}^{\leftrightarrow}(ba) \leq \mathrm{tr}^{\leftrightarrow} b$.

Proof (i). By Proposition 4.2(iii), $a(I)$ is positive matrix. Hence, by Proposition 4.4(iii), $\mathrm{tr}^{\leftrightarrow} a = \mathrm{tr}[a(I)] \geq 0$. (ii). By Proposition 4.2(iii) and 4.4(v), $\mathrm{tr}^{\leftrightarrow}(ab) = \mathrm{tr}[a^{\updownarrow}(I) b(I)] \leq \mathrm{tr}\, b(I) = \mathrm{tr}^{\leftrightarrow} b$. (iii) is shown in a sinilar way. $\square$

Definition 4.4 $a \in {}^{\star}\mathcal{L}(\mathbf{M})$ is called a *hyperfinite operation*, or simply an *operation* if $a^{\diamond}$ is positive and $a(I), a^{\updownarrow}(I) \leq I$.

Let Op denote the set of all operations. Op is an internal semigroup with involution $\updownarrow$ because if $a, b \in Op$, then $a^{\updownarrow\diamond}$ and $(ab)^{\diamond}$ are positive, and $a(b(I)) \leq a(I) \leq I$.

Proposition 4.7 $a \in {}^{\star}\mathcal{L}(\mathbf{M})$ is an operation iff there exist $M_1, \cdots, M_{\kappa} \in \mathbf{M}\, (\kappa \in {}^{\star}\mathbf{N})$ such that for all $A \in \mathbf{M}$,

$$a(A) = \sum_{k=1}^{\kappa} M_k A M_k^*$$

and

$$\sum_{k=1}^{\kappa} M_k^* M_k \leq I, \; \sum_{k=1}^{\kappa} M_k M_k^* \leq I.$$

Proof Suppose $a^\diamond$ is positive. By Proposition 4.3 and 4.5, there are $M_1, ..., M_\kappa \in \mathbf{M}\,(\kappa \in {}^\star\mathbf{N})$ such that

$$a(A) = \sum_{k=1}^{\kappa} M_k A M_k^*,$$

$$a^\updownarrow(A) = \sum_{k=1}^{\kappa} M_k^* A M_k.$$

Hence, $a(I) \leq I$ iff $\sum_{k=1}^{\kappa} M_k M_k^* \leq I$, and $a^\updownarrow(I) \leq I$ iff $\sum_{k=1}^{\kappa} M_k^* M_k \leq I$. $\square$

Definition 4.5 If $P \in \mathbf{M}$ is a projector, i.e., $P^* = P^2 = P$, $p : A \mapsto PAP$ is called a *projecting operation*. If $U \in \mathbf{M}$ is a unitary matrix, $u : A \mapsto UAU^*$ is called a *unitary operation*, and u^{-1} is defined as $u^{-1} : A \mapsto U^*AU$. The identity $1 \in Op$ is called a *unit operation* . The zero element of Op is written as 0, i.e., $0 : A \mapsto O$ for any matrix A. Operation a is called *trivial* if for any $b \in Op$, $\mathrm{tr}^\leftrightarrow(ab) = \mathrm{tr}^\leftrightarrow(ba) = \mathrm{tr}^\leftrightarrow b$.

Definition 4.6 Let a, b be operations and $b \neq 0$ (and hence $\mathrm{tr}^\leftrightarrow b > 0$).

$$\overleftarrow{P}(a|b) =^\circ \left(\frac{\mathrm{tr}^\leftrightarrow(ab)}{\mathrm{tr}^\leftrightarrow b}\right)$$

$$\overrightarrow{P}(a|b) =^\circ \left(\frac{\mathrm{tr}^\leftrightarrow(ba)}{\mathrm{tr}^\leftrightarrow b}\right)$$

$$P(a) =^\circ \left(\frac{\mathrm{tr}^\leftrightarrow a}{\nu}\right).$$

$\overleftarrow{P}$ and $\overrightarrow{P}$ are called *Bayesian conditional probability over Op.* P is called *Bayesian probability over Op.*

We see that if $P(b) \neq 0$, $\overleftarrow{P}(a|b) = P(ab)/P(b), \overrightarrow{P}(a|b) = P(ba)/P(b)$. Belinfante would call $\overleftarrow{P}$ *predictive probability*, and $\overrightarrow{P}$ *retrodictive probability*; $\overleftarrow{P}(a|b)$ represents the probability that a measuring operation a of yes-no type outputs "yes" at the instant after the observation that b output "yes", and $\overrightarrow{P}(a|b)$ represents the probability that a output "yes"at the instant before the observation that b outputs "yes". In the following, "$\leftarrow$" reeds "predictive" and "$\rightarrow$" reeds "retrodictive".

Proposition 4.8 $\overleftarrow{P}, \overrightarrow{P}$ and P have the following properties for any $a, b, c \in Op$.

(i) $0 \leq \overleftarrow{P}(a|b), \overrightarrow{P}(a|b), P(a) \leq 1$ if $b \neq 0$

(ii) $\overleftarrow{P}(ab|c) = \overleftarrow{P}(b|c)\overleftarrow{P}(a|bc)$ if $c, bc \neq 0$.

(iii) $\overrightarrow{P}(ab|c) = \overrightarrow{P}(a|c)\overrightarrow{P}(b|ca)$ if $c, ca \neq 0$

(iv) $\overleftarrow{P}(a+b|c) = \overleftarrow{P}(a|c) + \overleftarrow{P}(b|c)$ if $c \neq 0$

(v) $\overrightarrow{P}(a+b|c)=\overrightarrow{P}(a|c)=\overrightarrow{P}(b|c)$ if $c\neq 0$

(vi) $P(a+b)=P(a)+P(b)$.

Proof (i) follows from Proposition 4.6. (ii)–(vi) are evident from the definition. □

Theorem 4.9 (Bayes-type Theorem) Suppose $K\in\mathbf{N}, a_1,...,a_K, b\in Op$ and $\sum_{k=1}^K a_k=1$. If $P(b)\neq 0$ and $a_1,...,a_K\neq 0$, then

$$\text{(i)}\quad \overrightarrow{P}(a_j|b)=\frac{\overleftarrow{P}(b|a_j)P(a_j)}{\sum_{k=1}^{K}\overleftarrow{P}(b|a_k)P(a_k)}$$

$$\text{(ii)}\quad \overleftarrow{P}(a_j|b)=\frac{\overrightarrow{P}(b|a_j)P(a_j)}{\sum_{k=1}^{K}\overrightarrow{P}(b|a_k)P(a_k)}$$

Proof By Proposition 4.8. □

If $a\neq 0, \overleftarrow{P}(\cdot|a), \overrightarrow{P}(\cdot|a)$ and $P(\cdot)$ are finitely additive. However, they do not have σ-additivity. We also see that Theorem 4.9 does not hold if we let K be an infinite number (i.e.,$K\in{}^\star\mathbf{N}\setminus\mathbf{N}$). σ-additivity will be argued later.

Proposition 4.10

(i) $\overrightarrow{P}(1|a)+\overleftarrow{P}(1|a)=1$ if $a\neq 0$

(ii) $\overrightarrow{P}(0|a)=\overleftarrow{P}(0|a)=0$ if $a\neq 0$

Let $u\in Op$ be a unitary operation and suppose $b\neq 0$.

(iii) $\overleftarrow{P}(uau^{-1}|uau^{-1})=\overleftarrow{P}(au^{-1}|ub)=\overleftarrow{P}(ua|b)=\overleftarrow{P}(a|bu)=\overleftarrow{P}(a|b)$

(iv) $\overrightarrow{P}(uau^{-1}|ubu^{-1})=\overrightarrow{P}(ua|bu^{-1})=\overrightarrow{P}(au|b)=\overrightarrow{P}(a|ub)=\overrightarrow{P}(a|b)$.

From (i),(iii) and (iv), we see that $\overleftarrow{P}(a|u)=\overrightarrow{P}(a|u)=P(a)$ and $\overleftarrow{P}(u|d)=\overrightarrow{P}(u|b)=P(u)=1$. (iii) and (iv) are called the unitary invariance of $\overleftarrow{P}$ and $\overrightarrow{P}$.

Theorem 4.11 Suppose $a,b\in Op$ and $b\neq 0$

(i) $\overleftarrow{P}(a|b)=\overrightarrow{P}(a^{\updownarrow}|b^{\updownarrow})$

(ii) $\overrightarrow{P}(a|b)=\overleftarrow{P}(a^{\updownarrow}|b^{\updownarrow})$

(iii) $P(a)=P(a^{\updownarrow})$

Proof (i)–(ii) follow from the fact that if $a\in Op$ then $\mathrm{tr}^{\leftrightarrow}a=\mathrm{tr}^{\leftrightarrow}a^{\updownarrow}$. □

Corollary Suppose $a_m, b_n\in Op$ and ${a_m}^{\updownarrow}=a_m, {b_n}^{\updownarrow}=b_n\,(m=1,...,M, n=1,...,N; M,N\in{}^\star\mathbf{N})$.

(i) $\overleftarrow{P}(a_1 \cdots a_M | b_1 \cdots b_N) = \overrightarrow{P}(a_M \cdots a_1 | b_N \cdots b_1)$

(ii) $P(a_1 \cdots a_M) = P(a_M \cdots a_1)$.

where if M (resp.N) is infinite, $a_1 \cdots a_M$ (resp. $b_1 \cdots b_N$) denotes hyperfinite product of operations.

Operation $a_M \cdots a_1$ is interpreted as a time series of physical operations $a_1 \to a_2 \to \cdots \to a_M$. Thus, if ${a_m}^{\updownarrow} = a_m$ $(m = 1, ..., M)$ (e.g., $a_1, ..., a_M$ are projecting operations), we may interpret $a_1 \cdots a_M$ as the "time reversal" of $a_M \cdots a_1$ from this corollary. Generally, $a^{\updownarrow}$ is interpreted as a "time reversal" of a from Theorem 4.11, and hence the theorem shows that we can describe the time reversal symmetry including measuring processes in the framework of hyperfinite operations. Thus, this framework is expected to complete Belinfante's program, that is, "time-symmetric quantum theory" including quantum measurement theory.

5 Hyperfinite Instrument

Definition 5.1 Let $a_\xi \in Op$ $(\xi \in \mathcal{O}, \mathcal{O}$ is a hyperfinite set). If function $\mathcal{I} : \mathcal{O} \to Op$ is internal and $\sum_{\xi \in \mathcal{O}} \mathcal{I}(\xi)$ is trivial, $\mathcal{I}$ is called a *hyperfinite instrument*, or simply an *instrument*. Any element of $\mathcal{O}$ is called an *outcome* of $\mathcal{I}$.

Let $\mathcal{I} : \mathcal{O}_1 \to Op$ and $\mathcal{J} : \mathcal{O}_2 \to Op$ be instruments. The product $\mathcal{I}\mathcal{J} : \mathcal{O}_1 \times \mathcal{O}_2 \to Op$ is defined as

$$(\mathcal{I}\mathcal{J})(\xi, \zeta) = \mathcal{I}(\xi)\mathcal{J}(\zeta).$$

The product of two hyperfinite instruments is also a hyperfinite instrument, because

$$\mathrm{tr}^{\leftrightarrow}\left(\sum_{\xi \in \mathcal{O}_1, \zeta \in \mathcal{O}_2} \mathcal{I}(\xi)\mathcal{J}(\zeta)a\right) = \mathrm{tr}^{\leftrightarrow}\Big(\sum_{\xi \in \mathcal{O}_1} \mathcal{I}(\xi)\Big)\Big(\sum_{\zeta \in \mathcal{O}_2} \mathcal{J}(\zeta)\Big)a$$

$$= \mathrm{tr}^{\leftrightarrow}\left(\sum_{\zeta \in \mathcal{O}_2} \mathcal{J}(\zeta)a\right) = \mathrm{tr}^{\leftrightarrow} a.$$

Let $\mathcal{I} : \mathcal{O} \to Op$ be a hyperfinite instrument and $a \in Op, a \neq 0$. If A is an internal subset of $\mathcal{O}$, then $\sum_{\alpha \in A} \mathcal{I}(\alpha)$ exists. We define $\overleftarrow{P}_{\mathcal{I}}, \overrightarrow{P}_{\mathcal{I}}$ and $P_{\mathcal{I}}$ as follows:

$$\overleftarrow{P}_{\mathcal{I}}(A|a) = \overleftarrow{P}(\sum_{\alpha \in A} \mathcal{I}(\alpha)|a).$$

$$\overrightarrow{P}_{\mathcal{I}}(A|a) = \overrightarrow{P}(\sum_{\alpha \in A} \mathcal{I}(\alpha)|a).$$

$$P_{\mathcal{I}}(A) = \overleftarrow{P}(A|1) = \overrightarrow{P}_{\mathcal{I}}(A|1) = P(\sum_{\alpha \in A} \mathcal{I}(\alpha)).$$

$\mathcal{P}^i(\mathcal{O})$, the set of all the internal subsets of $\mathcal{O}$, is a finitely additive family, and it is shown that $(\mathcal{O}, \mathcal{P}^i(\mathcal{O}), \overleftarrow{P}_{\mathcal{I}} (\cdot|a))$ is a completely additive probability space. By Hopf's extension theorem, there exists σ-additive probability space $(\mathcal{O}, \sigma\mathcal{P}^i(\mathcal{O}), \overleftarrow{P}_{\mathcal{I}} (\cdot|a))$ which is the extension of $(\mathcal{O}, \mathcal{P}^i(\mathcal{O}), \overleftarrow{P} (\cdot|a))$, where $\sigma\mathcal{P}^i(\mathcal{O})$ is the least σ- field of sets greater than $\mathcal{P}^i(\mathcal{O})$. Let $(\mathcal{O}, L\mathcal{P}^i(\mathcal{O}), L\overleftarrow{P}_{\mathcal{I}} (\cdot|a))$ denote the Lebesgue completion of $(\mathcal{O}, \sigma\mathcal{P}^i(\mathcal{O}), \overleftarrow{P}_{\mathcal{I}} (\cdot|a))$, and $\overleftarrow{L} (\mathcal{I}|a)$ be the abbreviation of it. $\overrightarrow{L} (\mathcal{I}|a) = (\mathcal{O}, L\mathcal{P}^i(\mathcal{O}), L\overrightarrow{P}_{\mathcal{I}} (\cdot|a))$ is defined in a similar way. Let $L(\mathcal{I}) = (\mathcal{O}, L\mathcal{P}^i(\mathcal{O}), LP_{\mathcal{I}}(\cdot)) = \overrightarrow{L} (\mathcal{I}|1) = \overleftarrow{L} (\mathcal{I}|1)$.

$\overrightarrow{L} (\mathcal{I}|a), \overleftarrow{L} (\mathcal{I}|a)$ and $L(\mathcal{I})$ are called the *Loeb probability space generated by* $\mathcal{I}$ (general theory of Loeb space and its application are seen in, e.g., Ref.(13)).

Bayesian conditional probability in terms of instruments is defined as follows. Let $\mathcal{I} : \mathcal{O}_1 \to Op$ and $\mathcal{J} : \mathcal{O}_2 \to Op$ be instruments. If $A \in L\mathcal{P}^i(\mathcal{O}_1), B \in L\mathcal{P}^i(\mathcal{O}_2)$ and $LP_{\mathcal{J}}(B) > 0$,

$$\overleftarrow{P}_{\mathcal{I},\mathcal{J}} (A|B) = LP_{\mathcal{I},\mathcal{J}}(A \times B)/LP_{\mathcal{J}}(B),$$

$$\overrightarrow{P}_{\mathcal{I},\mathcal{J}} (A|B) = LP_{\mathcal{J},\mathcal{I}}(B \times A)/LP_{\mathcal{J}}(B),$$

These are well-defined because it is known that if $A \in L\mathcal{P}^i(\mathcal{O}_1)$ and $B \in L\mathcal{P}^i(\mathcal{O}_2)$ then $A \times B \in L\mathcal{P}^i(\mathcal{O}_1 \times \mathcal{O}_2)$ and $B \times A \in L\mathcal{P}^i(\mathcal{O}_2 \times \mathcal{O}_1)$.

Note that while $\overleftarrow{P}_{\mathcal{I}} (A|a)$ and $\overrightarrow{P}_{\mathcal{I}} (A|a)$ are defined if $a \neq 0$ (even if the (unconditional) Bayesian probability $P(a)$ equals 0), $\overleftarrow{P}_{\mathcal{I},\mathcal{J}} (A|B)$ and $\overrightarrow{P}_{\mathcal{I},\mathcal{J}} (A|B)$ are defined only if $LP_{\mathcal{I}}(B) > 0$.

6 Bayesian State

In Bayesian statistics, one of the most foundational concept is that of a priori distribution, which is understood as the probability distribution in which we have the least information or knowledge about a set of objects, that is, a sample space. In the following, we argue the concept of quantum Bayesian a priori and a posteriori states in terms of operations and instruments. The ideas are as follows. At an actual measurement process, or generally, at an experimental operation process, we often do not know the initial (resp. final) state of objects. In such a case, we do not predict (resp. retrodict) the outcome from the initial (resp. final) state, but, conversely, guess the initial (resp. final) state from the outcome. The Bayesian a priori (resp. a posteriori) state of the process is the initial (resp. final) state that we guess in such a way.

The definition of the concept uses the following usual definition of states over C^*- algebras[(2)].

Definition 6.1 Let $\mathcal{A}$ be a C^*-algebra. We define the norm of a functional f over $\mathcal{A}$ by

$$\| f \| = \sup\{|f(A)| : \| A \| = 1\}$$

A linear functional ω over $\mathcal{A}$ is defined to be positive if

$$\omega(A^* A) \geq 0$$

for all $A \in \mathcal{A}$. A positive linear functional ω over $\mathcal{A}$ with $\| \omega \| = 1$ is called a *state*.

It is shown that if $\mathcal{A}$ contains a unit $1_{\mathcal{A}}$, then $\| \omega \| = 1$ iff $\omega(1_{\mathcal{A}}) = 1$ (see Ref.(14),p.49).

Let $\mathcal{A} = \hat{\mathbf{M}}_\infty = \mathrm{fin}_\infty(\mathbf{M})/\mu_\infty(0)$. $(\hat{\mathbf{M}}_\infty, ^\circ \| \cdot \|_\infty)$ is a C^*-algebra as we have already seen at section 2.

Definition 6.2 Let $a \in Op$ and $a \neq 0$. States $\overleftarrow{\omega}_a$ and $\overrightarrow{\omega}_a$ over $\hat{\mathbf{M}}_\infty$ are defined as follows:

$$\overrightarrow{\omega}_a (\hat{A}) = ^\circ \left(\frac{\mathrm{tr}\, a(A)}{\mathrm{tr} \overleftrightarrow{a}} \right)$$

$$\overleftarrow{\omega}_a (\hat{A}) = ^\circ \overline{\left(\frac{\mathrm{tr}\, a^{\updownarrow}(A^*)}{\mathrm{tr} \overleftrightarrow{a}} \right)}$$

where $\hat{A} = A + \mu_\infty(0)$. $\overrightarrow{\omega}_a$ and $\overleftarrow{\omega}_a$ are called *Bayesian a priori state* and *Bayesian a posteriori state of* a, respectively. That $\overrightarrow{\omega}_a$ and $\overleftarrow{\omega}_a$ are well-defined is shown as follows.

Suppose $A, B \in \mathrm{fin}_\infty(\mathbf{M})$ and $A - B \in \mu_\infty$. Notice that $\mathrm{tr}[a(A)] = \overline{\mathrm{tr}[A^* a^{\updownarrow}(I)]}$ and $\overline{\mathrm{tr}[a^{\updownarrow}(A^*)]} = \mathrm{tr}[Aa(I)]$. Hence,

$$\frac{\mathrm{tr}[a(A)]}{\mathrm{tr} \overleftrightarrow{a}} - \frac{\mathrm{tr}[a(B)]}{\mathrm{tr} \overleftrightarrow{a}} = \frac{\mathrm{tr}[a(A-B)]}{\mathrm{tr}[a^{\updownarrow}(I)]} = \mathrm{tr} \left(\frac{(A-B)^* a^{\updownarrow}(I)}{\mathrm{tr}[a^{\updownarrow}(I)]} \right).$$

$a^{\updownarrow}(I)/\,\mathrm{tr}[a^{\updownarrow}(I)]$ is positive and its trace is one, that is, $a^{\updownarrow}(I)/\,\mathrm{tr}[a^{\updownarrow}(I)]$ is a density matrix. Let $|\xi_1\rangle, ..., |\xi_\nu\rangle$ be the normalized eigenvectors of $a^{\updownarrow}(I)/\,\mathrm{tr}[a^{\updownarrow}(I)]$, and $\lambda_1, ..., \lambda_\nu$ be the eigenvalues which correspond to them, respectively. $|\xi_1\rangle, ..., |\xi_\nu\rangle$ is a complete orthonormal system of ${}^\star\mathbf{C}^*$. Because $(A-B)^* \in \mu_\infty(0)$,$\lambda_k \geq 0$ and $\sum_{k=1}^{\nu} \lambda_k = 1$,

$$| \,\mathrm{tr}(A-B)^* a^{\updownarrow}(I)/\,\mathrm{tr}[a^{\updownarrow}(I)]|$$

$$= | \sum_{k=1}^{\nu} \langle \xi_k | (A-B)^* \frac{a^{\updownarrow}(I)}{\mathrm{tr}[a^{\updownarrow}(I)]} | \xi_k \rangle |$$

$$\leq \sum_{k=1}^{\nu} \lambda_k | \langle \xi_k | (A-B)^* | \xi_k \rangle | \approx 0.$$

Therefore, $\overrightarrow{\omega}_a (\hat{A}) = \overrightarrow{\omega}_a (\hat{B})$. $\overleftarrow{\omega}_a (\hat{A}) = \overleftarrow{\omega}_a (\hat{B})$ is shown in a similar fashion.

Definition 6.3 Let $\mathcal{I} : \mathcal{O} \to Op$ be a hyperfinite instrument and S an internal subset of $\mathcal{O}$. States $\overrightarrow{\omega}_{\mathcal{I},S}$ and $\overleftarrow{\omega}_{\mathcal{I},S}$ over $\hat{\mathbf{M}}_\infty$ are defined as follows:

$$\overrightarrow{\omega}_{\mathcal{I},S}(\hat{A}) = {}^\circ\left(\frac{\sum_{\alpha \in S} \mathrm{tr}[\mathcal{I}(\alpha)(A)]}{\sum_{\alpha \in S} \mathrm{tr}\,{}^{\leftrightarrow}\mathcal{I}(\alpha)}\right)$$

$$\overleftarrow{\omega}_{\mathcal{I},S}(\hat{A}) = {}^\circ\overline{\left(\frac{\sum_{\alpha \in S} \mathrm{tr}[\mathcal{I}(\alpha)^{\updownarrow}(A^*)]}{\sum_{\alpha \in S} \mathrm{tr}\,{}^{\leftrightarrow}\mathcal{I}(\alpha)}\right)}$$

$\overrightarrow{\omega}_{\mathcal{I},S}$ ($\overleftarrow{\omega}_{\mathcal{I},S}$) is called *Bayesian a priori (a posteriori) state of* $\mathcal{I}$ when its outcome is in S.

Note that if $a = \sum_{\alpha \in S} \mathcal{I}(\alpha)$, then $\overrightarrow{\omega}_{\mathcal{I},S}(\hat{A}) = \overrightarrow{\omega}_a(\hat{A}), \overleftarrow{\omega}_{\mathcal{I},S}(\hat{A}) = \overleftarrow{\omega}_a(\hat{A})$, and hence $\overrightarrow{\omega}_{\mathcal{I},S}$ and $\overleftarrow{\omega}_{\mathcal{I},S}$ are well-defined.

We have following relations concerning $\overleftarrow{P}, \overrightarrow{P}, \overleftarrow{\omega}, \overrightarrow{\omega}$. Let $a \in Op$ satisfy $a(A) = \sum_{k=1}^{\kappa} M_k A M_k^*$ for all $A \in \mathbf{M}$ where $\kappa \in {}^\star\mathbf{N}$, $M_1, ..., M_\kappa \in \mathbf{M}$ and $\sum_{k=1}^{\kappa} M_k^* M_k \leq I$, $\sum_{k=1}^{\kappa} M_k M_k^* \leq I$. Let $b \in Op$ and $b \neq 0$. We have

$$\overleftarrow{P}(a|b) = \overleftarrow{\omega}_b(\hat{M})$$

$$\overrightarrow{P}(a|b) = \overrightarrow{\omega}_b(\hat{M}')$$

where $M = \sum_{k=1}^{\kappa} M_k^* M_k$, $M' = \sum_{k=1}^{\kappa} M_k M_k^*$.

7 Remarks and Problems

1. This paper presents no content but a framework of time-symmetric quantum physics. To describe the physical content in the framework, we first need representations of canonical commutation relation on hyperfinite-dimensional linear spaces. Ojima and Ozawa[15] started the research in this direction.

2. Some mathematical problems are left. First, the condition of the definition of $\overleftarrow{P}_{\mathcal{I},\mathcal{J}}$ and $\overrightarrow{P}_{\mathcal{I},\mathcal{J}}$, that is, $LP_{\mathcal{J}}(B) > 0$ should be weakened. $\overleftarrow{P}_{\mathcal{I},\mathcal{J}}(A|B)$ and $\overrightarrow{P}_{\mathcal{I},\mathcal{J}}(A|B)$ seem not to be always meaningless even if $LP_{\mathcal{J}}(B) = 0$. Secondly, Bayesian state of an instrument is defined only when S is internal in Definition 6.3. Can we extend the definition so that it contains the cases that S is external? Thirdly, can we define Bayesian a priori and a posteriori states as normal states on von Neumann algebra $\hat{\mathbf{M}}$?

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